



# Improving the NSGA-II Performance with an External Population

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# Presentation Plan

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- Introduction:
  - multiobjective optimization
  - NSGA-II
- The NSGA-II-EXT algorithm
  - overview
  - external population merging methods
- Experiments
- Results
- Conclusion



# Multiobjective Optimization (1/3)

- More than one objective function

Formally:

$$\max_{x \in \Omega} F(x) = [f_1(x), f_2(x), \dots, f_m(x)]^T$$

$$F : \Omega \rightarrow R^m$$

where:

$\Omega$  – search space

$R^m$  – objective space

$f_i$  – objective functions

- Usually no single solution optimizes all criteria
- Two solutions are not always comparable



# Multiobjective Optimization (2/3)

- Domination relation

$x_1, x_2 \in \Omega$  – solutions

$u = F(x_1), v = F(x_2)$

Solution  $x_1$  **dominates**  $x_2$  ( $x_1 \succ x_2$ ) iff:

$$\forall_{i \in \{1, 2, \dots, m\}} \cdot u_i \geq v_i$$

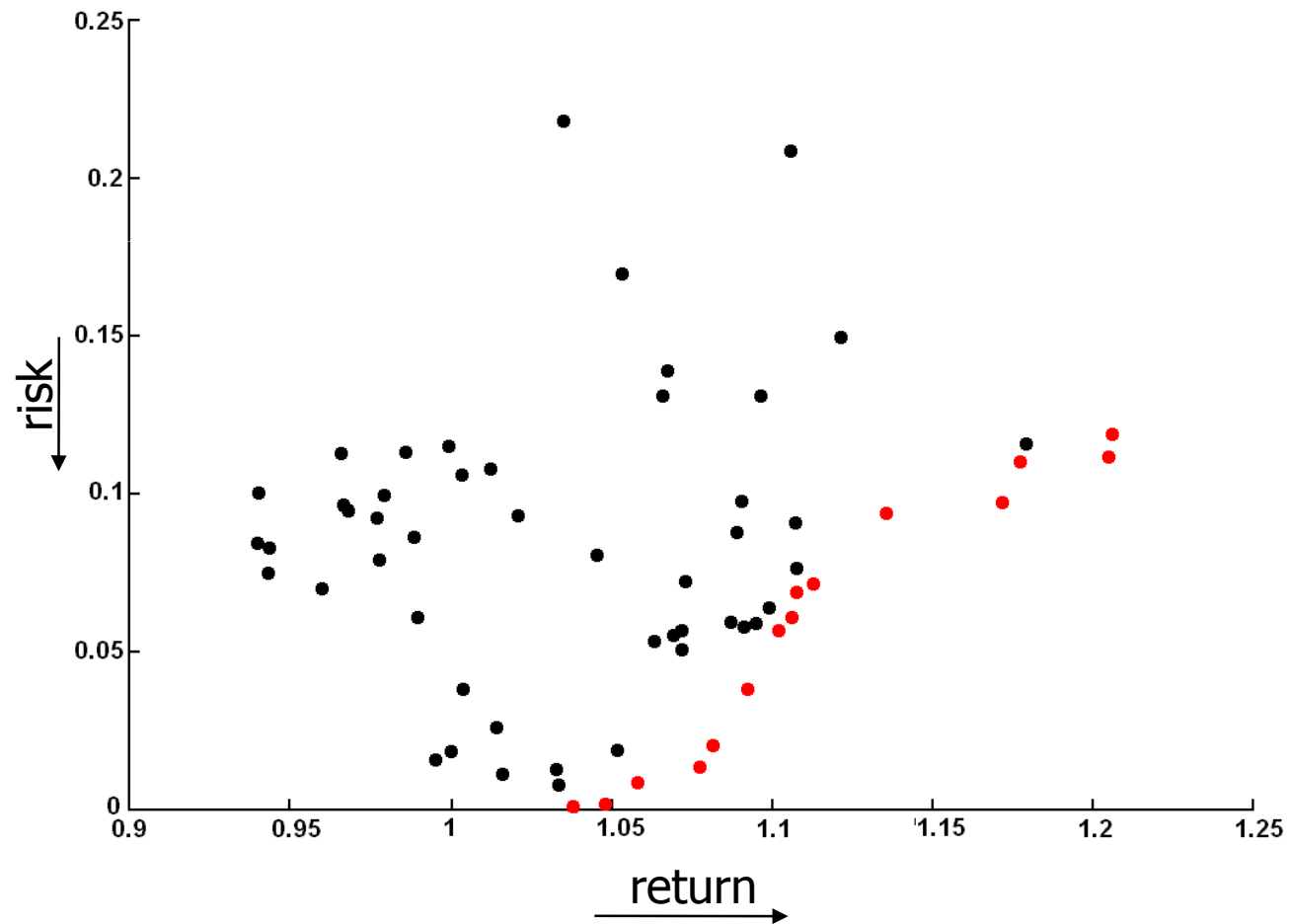
$$\exists_{i \in \{1, 2, \dots, m\}} \cdot u_i > v_i$$

- Instead of looking for one „the best“ solution we try to obtain a Pareto Front of non-dominated solutions

# Multiobjective Optimization (3/3)

Many real-life problems are multiobjective

Example: max. investment return, min. risk

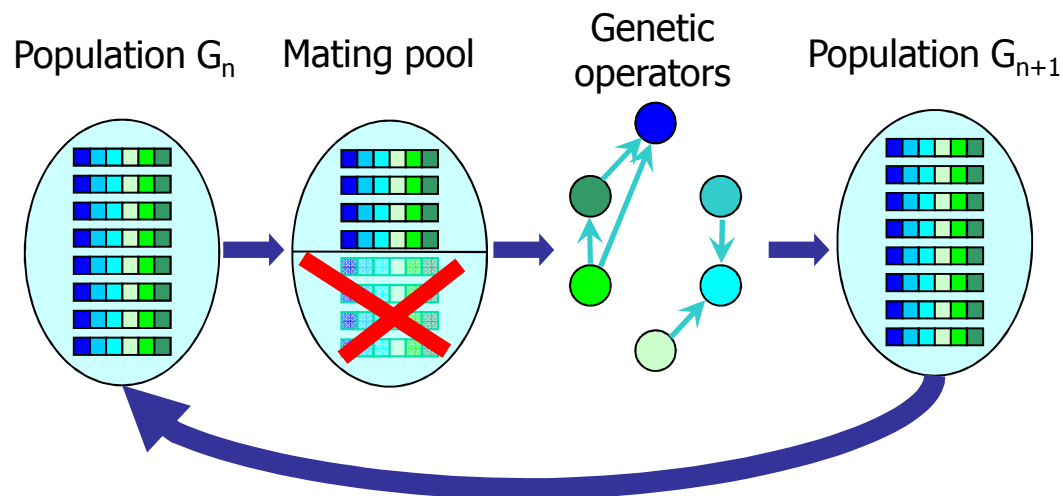


# Evolutionary algorithms

Population-based search

A specimen represents a solution

Evaluation of a specimen  $x$ :  $f(x) \in R$



# NSGA-II (1/3)

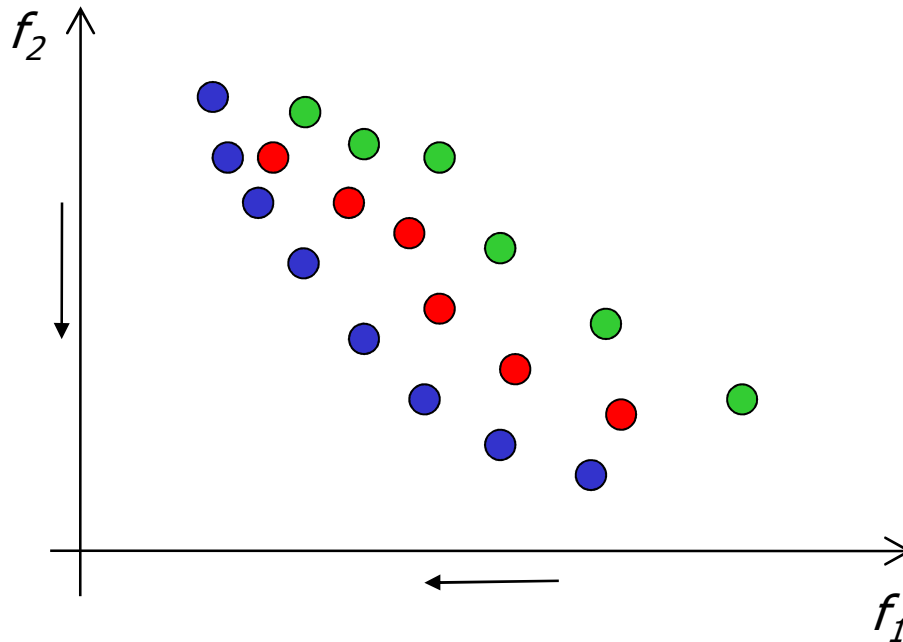
## Pareto Front ranking

$FR_0$  = non-dominated specimens from the population  $P$

$FR_1$  = non-dominated specimens from  $P \setminus FR_0$

$FR_2$  = non-dominated specimens from  $P \setminus (FR_0 \cup FR_1)$

$FR_3 = \dots$





## NSGA-II (2/3)

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Crowding distance (calculated for each front  $FR_i$  separately)

$$n = |FR_i|$$

for each specimen  $x \in FR_i$

$$\delta_x = 0$$

for each objective number  $m$  sort specimens w.r.t.  $f_m$

$$I = \text{sort}(FR_i, m)$$

specimens with the smallest and largest values of  $f_m$

$$\delta_{I[1]} = \infty$$

$$\delta_{I[n]} = \infty$$

for  $k = 2, \dots, n - 1$

$$\delta_{I[k]} = \delta_{I[k]} + \frac{f_m(I[k+1]) - f_m(I[k-1])}{\max_{x \in FR_i}(f_m(x)) - \min_{x \in FR_i}(f_m(x))}$$





## NSGA-II (3/3)

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- Mating pool selection:
  - Binary tournament
  - Lower front number wins
  - If both in the same  $FR_i$ , then greater crowding distance wins
- Problem-specific operators (e.g. crossover, mutation)
- Offspring join the current population
- Elitist reduction to original population size
  - Fronts in order:  $FR_0, FR_1, \dots$
  - If the last  $FR_i$  considered does not fit, then specimens with the highest crowding distance are selected



## NSGA-II-EXT (1/2)

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- External population  $EP$  is used (nondominated specimens found so far)

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**Algorithm 1.** The NSGA-II-EXT algorithm.

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```
 $P = \text{InitPopulation}()$   
 $EP = \emptyset$   
 $\text{Evaluate}(P)$   
 $\text{Rank}(P)$   
for  $t = 2 \rightarrow N_{Gen}$  do  
     $P_{parent} = \text{BinaryTournament}(P)$   
     $P_{offspring} = \text{GenerateOffspring}(P_{parent})$   
     $\text{Evaluate}(P_{offspring})$   
     $P = P \cup P_{offspring}$   
  
     $EP = \text{AddToEP}(EP, P)$   
     $P = \text{MergeEP}(P, EP)$   
     $P = \text{Reduce}(P)$   
end for
```

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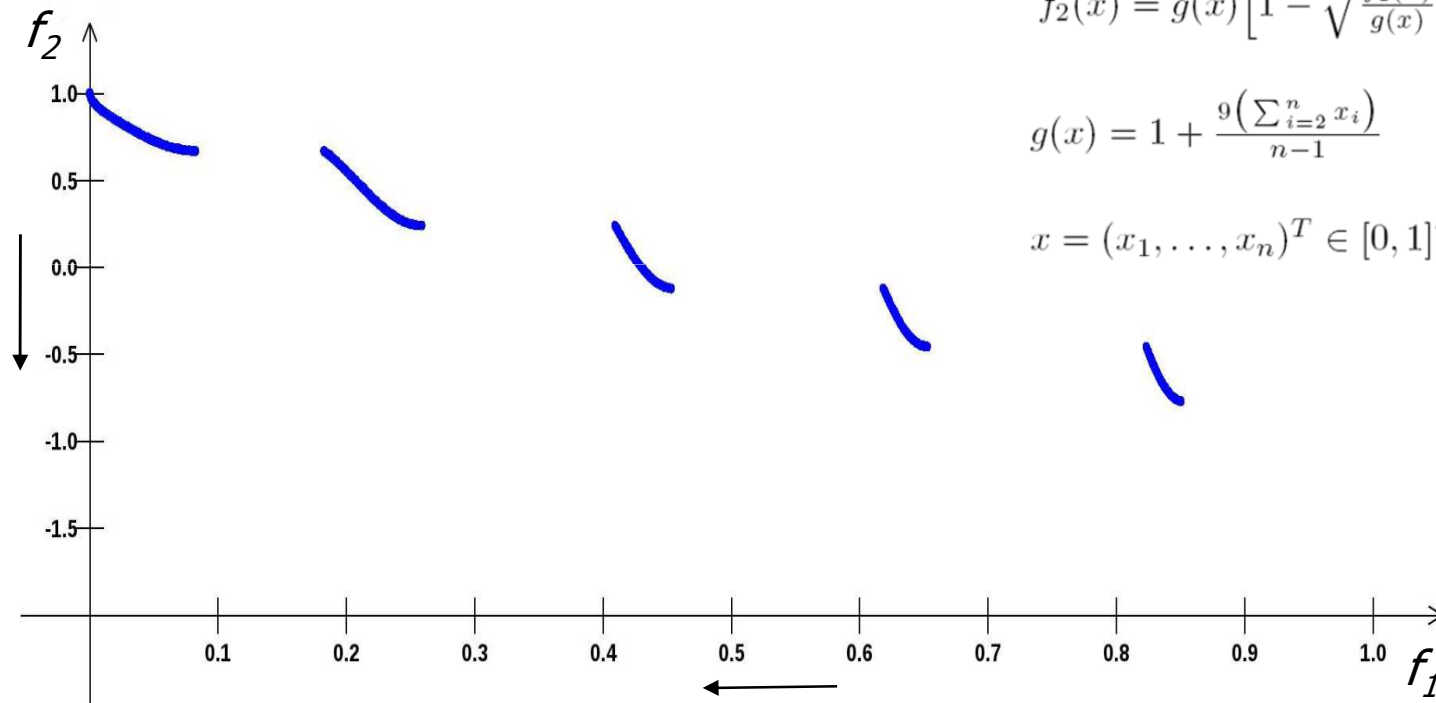
## NSGA-II-EXT (2/2)

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- **MergeEP** – reintroduces specimens from the external population  $EP$  back to the main population  $P$
- Reintroduction procedures were tested in this paper:
  - **none** - no  $EP$  merging. This is equivalent to the regular NSGA-II algorithm,
  - **all** - all specimens from  $EP$  are added to  $P$ ,
  - **rand** -  $q\%$  of specimens from  $EP$  selected at random are added to  $P$ ,
  - **cd** – crowding distance (CD) is calculated for specimens in  $P \cup EP$ , then  $q\%$  of specimens from  $EP$  with the largest CD values are added to  $P$

# Experiments

## ■ ZDT Problem suite<sup>[1]</sup>



ZDT-3

$$f_1(x) = x_1$$

$$f_2(x) = g(x) \left[ 1 - \sqrt{\frac{f_1(x)}{g(x)}} - \frac{f_1(x)}{g(x)} \sin(10\pi x_1) \right]$$

$$g(x) = 1 + \frac{9 \left( \sum_{i=2}^n x_i \right)}{n-1}$$

$$x = (x_1, \dots, x_n)^T \in [0, 1]^n, n = 30.$$

[1] E. Zitzler, K. Deb, L. Thiele, „ Comparison of Multiobjective Evolutionary Algorithms: Empirical Results”, Evolutionary Computation 2, vol. 8, pp. 173-195, MIT Press , 2000



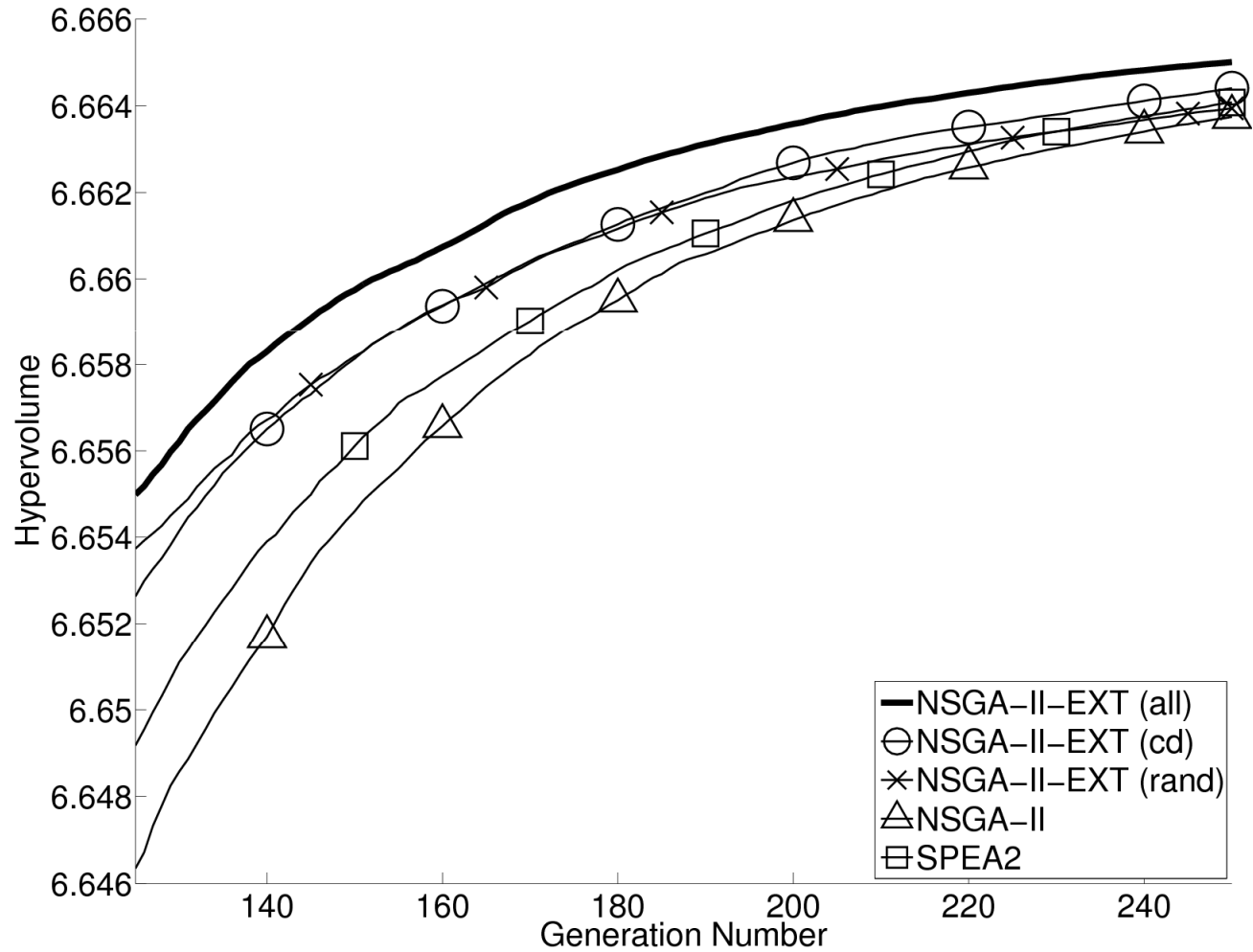
# Experiments

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- **Operators:** SBX and polynomial mutation
- **Parameters:**
  - $P_{cross} = 0.9$  and  $P_{mut} = 0.033$
  - $N_{pop} = 100$  and  $N_{gen} = 250$
  - Distribution index values:  $\eta_{cross} = 20$  and  $\eta_{mut} = 20$
  - $q = 75\%$  (fraction of  $EP$  reintroduced to  $P$ )
- Quality measure: **hypervolume (higher = better)**
  - calculated for  $EP$  obtained after each generation
  - common reference point
- Median values calculated from **30 runs**

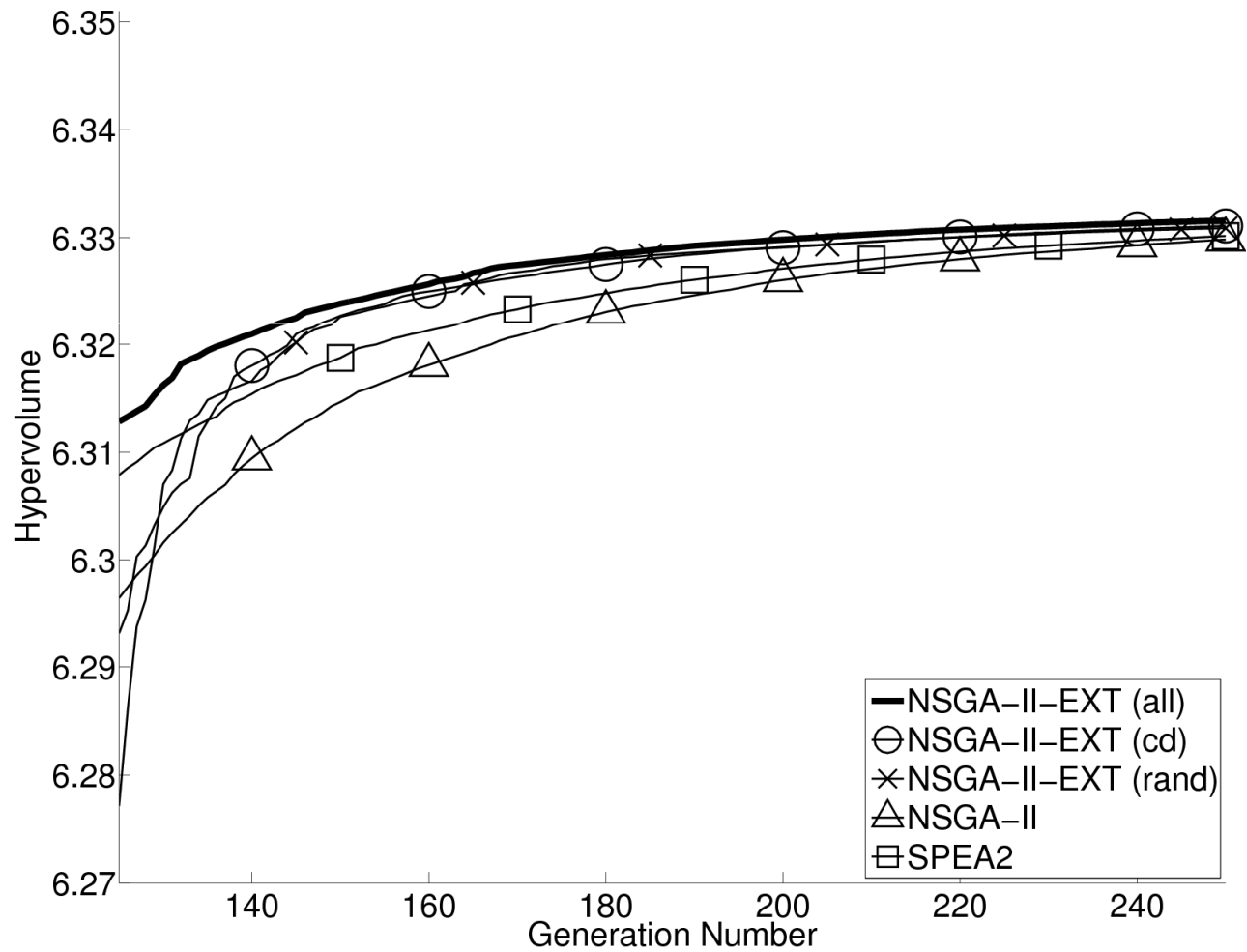
# Results

## ZDT-1



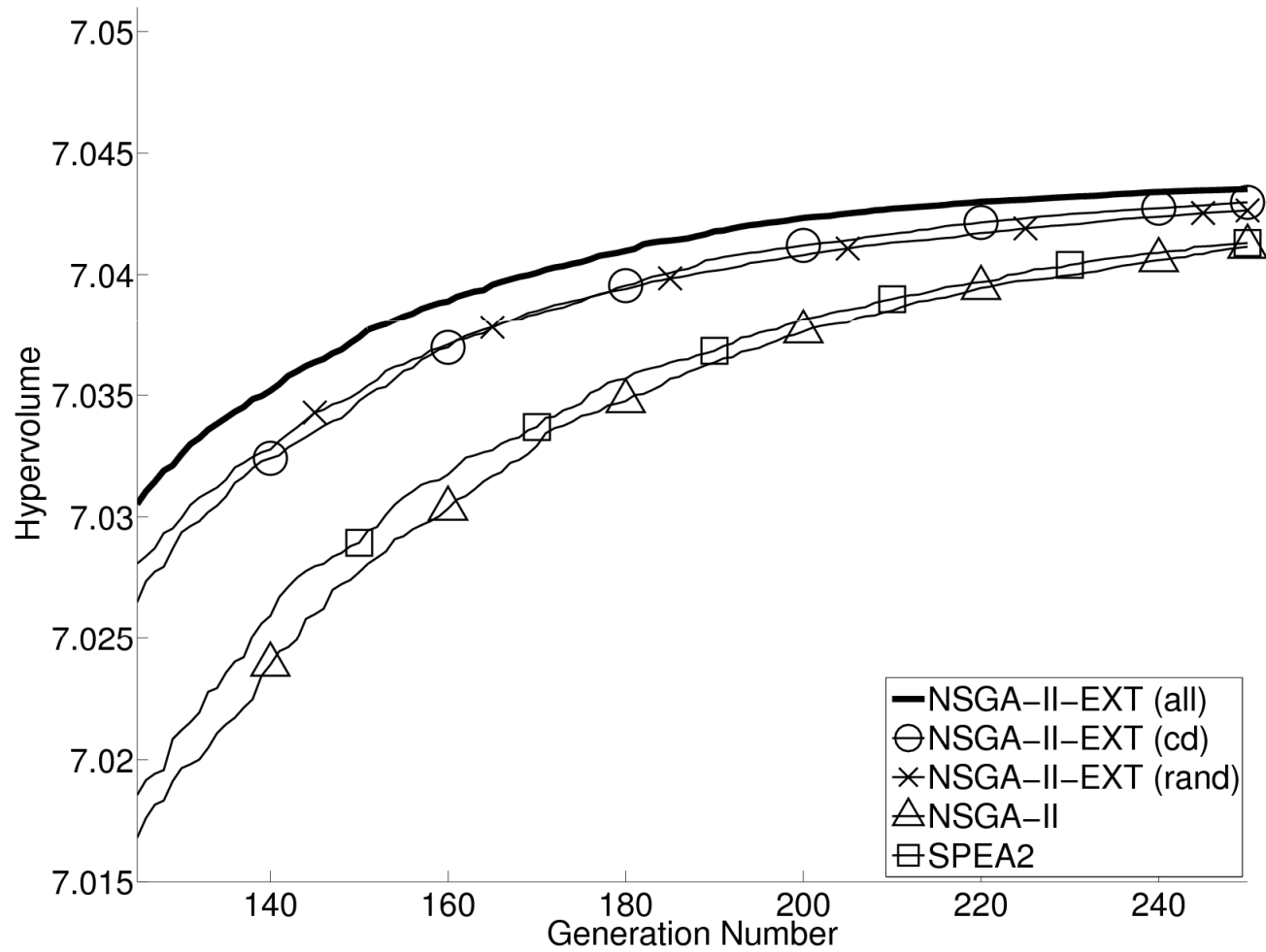
# Results

## ZDT-2



# Results

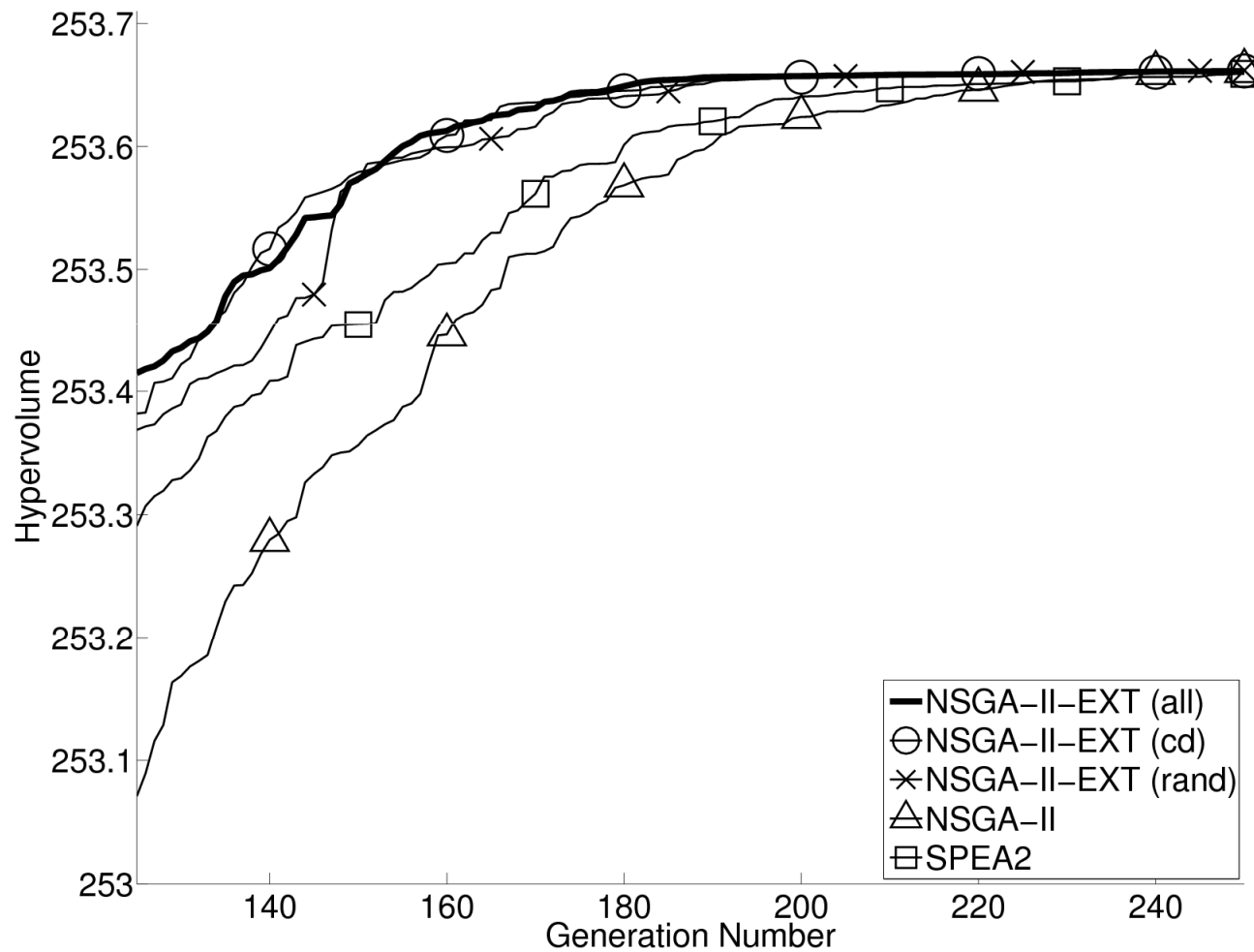
## ZDT-3





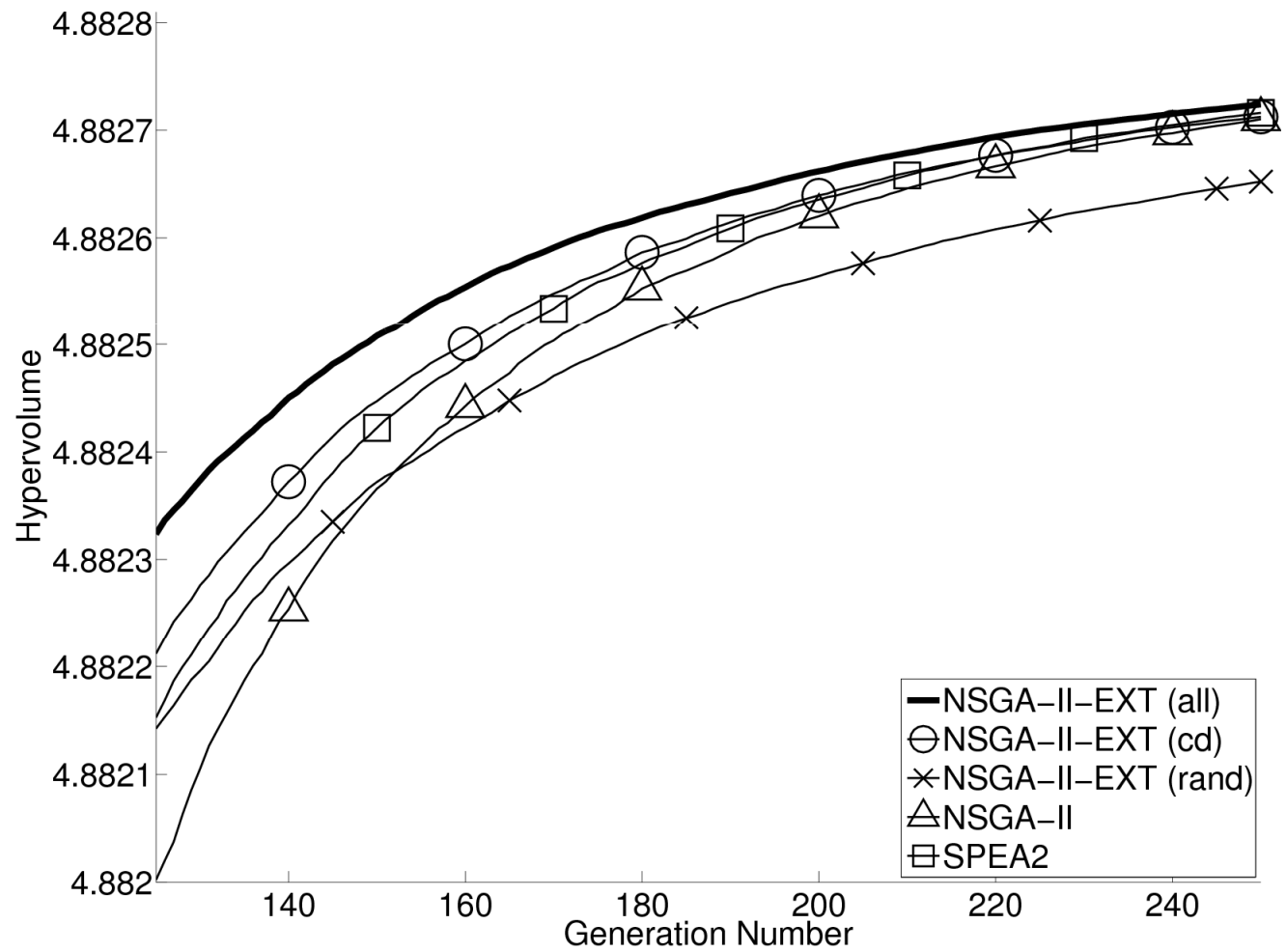
# Results

## ZDT-4



# Results

## ZDT-6





# Results

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- Statistical verification
  - Two compared algorithms produce populations of solutions
  - Hypervolume values calculated using a common reference point
  - Null hypothesis: two distributions have equal medians
- Values of the hypervolume obtained at the 250th generation were used for comparison with NSGA-II and SPEA2



# Results

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- Interpretation
  - **significant:** median hypervolume not lower than the median for „compared to“ algorithm and  $p\text{-value} \leq 0.05$
  - **insignificant:** median hypervolume not lower than the median for „compared to“ algorithm and  $p\text{-value} > 0.05$
  - **worse:** median hypervolume lower than the median for „compared to“ algorithm



# Results

## ZDT-1

Algorithm	median	std	compared to NSGA-II		compared to SPEA-2	
			p-value	interp.	p-value	interp.
NSGA-II	6.6637	0.00027870	—		0.00017317	worse
NSGA-II-EXT (cd)	6.6644	0.00025231	2.7927e-009	significant	7.4755e-006	significant
NSGA-II-EXT (rand)	6.6639	0.00033312	0.12057	insignificant	0.041325	worse
NSGA-II-EXT (all)	6.6650	0.00022047	2.8719e-011	significant	4.2855e-011	significant
SPEA-2	6.6641	0.00021345	0.00017317	significant	—	

## ZDT-2

Algorithm	median	std	compared to NSGA-II		compared to SPEA-2	
			p-value	interp.	p-value	interp.
NSGA-II	6.3298	0.00049425	—		0.00010095	worse
NSGA-II-EXT (cd)	6.3311	0.0004308	3.8787e-011	significant	7.4386e-009	significant
NSGA-II-EXT (rand)	6.3309	0.00041632	9.4449e-011	significant	4.9898e-007	significant
NSGA-II-EXT (all)	6.3316	0.00031698	2.8719e-011	significant	5.2283e-011	significant
SPEA-2	6.3301	0.00041895	0.00010095	significant	—	



# Results

## ZDT-3

Algorithm	median	std	compared to NSGA-II		compared to SPEA-2	
			p-value	interp.	p-value	interp.
NSGA-II	7.0412	0.00040727	—		0.34404	worse
NSGA-II-EXT (cd)	7.0430	0.015934	1.0241e-007	significant	6.2639e-008	significant
NSGA-II-EXT (rand)	7.0426	0.015917	1.0241e-007	significant	6.2639e-008	significant
NSGA-II-EXT (all)	7.0435	0.0095554	5.3167e-010	significant	4.403e-010	significant
SPEA-2	7.0413	0.013224	0.34404	insignificant	—	

## ZDT-4

Algorithm	median	std	compared to NSGA-II		compared to SPEA-2	
			p-value	interp.	p-value	interp.
NSGA-II	6.6613	0.97866	—		0.30071	insignificant
NSGA-II-EXT (cd)	6.6614	1.5487	0.56421	insignificant	0.086346	insignificant
NSGA-II-EXT (rand)	6.6618	0.031004	0.28711	insignificant	0.017299	significant
NSGA-II-EXT (all)	6.6613	0.031059	0.47792	worse	0.028663	significant
SPEA-2	6.6590	0.037741	0.30071	worse	—	



# Results

## ZDT-6

Algorithm	median	std	compared to NSGA-II		compared to SPEA-2	
			p-value	interp.	p-value	interp.
NSGA-II	4.8827	1.0912e-005	—		0.0025611	worse
NSGA-II-EXT (cd)	4.8827	9.9921e-006	0.58436	insignificant	0.0084977	worse
NSGA-II-EXT (rand)	4.8827	1.5931e-005	3.5098e-011	worse	2.8719e-011	worse
NSGA-II-EXT (all)	4.8827	8.4028e-006	2.2902e-008	significant	0.0010302	significant
SPEA-2	4.8827	9.4905e-006	0.0025611	significant	—	

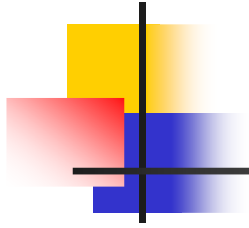


## Conclusion

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- NSGA-II-EXT stores all nondominated solutions found so far in an **external population** *EP*
- Several methods of **merging of specimens from** *EP* **with the population** were tested
- The **best performing variant**: NSGA-II-EXT (all) algorithm (merging of the entire external population)
  - For all tests problems except ZDT-4 the NSGA-II-EXT (all) outperformed the NSGA-II algorithm
  - The SPEA-2 algorithm was outperformed by NSGA-II-EXT (all) in all the tests
  - Both results are statistically significant





Thank you!  
(questions?)